Contents

1 DFT_Gates_def_PROB Theory

Built: 19 July 2018

Parent Theories: integration_before_after, PIE_EXTREAL

1.1 Definitions

[All_distinct_events_def]

```
⊢ ∀p L t.
      All_distinct_events p L t ⇔
      ALL_DISTINCT (MAP (λa. DFT_event p a t) L)
```

[ALWAYS_def]

```
⊢ ALWAYS = (λs. 0)
```

[CSP_def]

```
⊢ ∀Y X. CSP Y X = (λs. if Y s < X s then X s else PosInf)
```

[D_AND_def]

```
⊢ ∀X Y. D_AND X Y = (λs. max (X s) (Y s))
```

[D_BEFORE_def]

```
⊢ ∀X Y. D_BEFORE X Y = (λs. if X s < Y s then X s else PosInf)
```

[D_INCLUSIVE_BEFORE_def]

```
⊢ ∀X Y.
      D_INCLUSIVE_BEFORE X Y =
      (λs. if X s ≤ Y s then X s else PosInf)
```

[D_OR_def]

```
⊢ ∀X Y. D_OR X Y = (λs. min (X s) (Y s))
```

[D_SIMULT_def]

```
⊢ ∀X Y. D_SIMULT X Y = (λs. if X s = Y s then X s else PosInf)
```

[DFT_event_def]

```
⊢ ∀p X t. DFT_event p X t = {s | X s ≤ Normal t} ∩ p_space p
```

[FDEP_def]

```
⊢ ∀X Tr. FDEP X Tr = (λs. min (X s) (Tr s))
```

[HSP_def]

```
⊢ ∀Y X. HSP Y X = (λs. max (Y s) (X s))
```

[NEVER_def]

```
⊢ NEVER = (λ s. PosInf)
```

[NEVER_events_def]

```
⊢ ∀ X Y. NEVER_events X Y ⇔ (D_AND X Y = NEVER)
```

[P_AND_def]

```
⊢ ∀ X Y. P_AND X Y = (λ s. if X s ≤ Y s then Y s else PosInf)
```

[rv_gt0_ninfty_def]

```
⊢ (rv_gt0_ninfty [] ⇔ T) ∧
  ∀ h t.
      rv_gt0_ninfty (h::t) ⇔
      (∀ s. 0 ≤ h s ∧ h s ≠ PosInf) ∧ rv_gt0_ninfty t
```

[shared_spare_def]

```
⊢ ∀ X Y Z_a Z_d.
      shared_spare X Y Z_a Z_d =
      D_OR
        (D_OR (D_AND X (D_BEFORE Z_d X))
           (D_AND X (D_BEFORE Y X)))
        (D_AND Z_a (D_BEFORE X Z_a))
```

[UNIONL_def]

```
⊢ (UNIONL [] = { }) ∧ ∀ s ss. UNIONL (s::ss) = s ∪ UNIONL ss
```

[WSP_def]

```
⊢ ∀ Y X_a X_d.
      WSP Y X_a X_d =
      D_OR
        (D_OR
           (D_OR (D_AND X_a (D_BEFORE Y X_a))
              (D_AND Y (D_BEFORE X_d Y))) (D_SIMULT Y X_a))
        (D_SIMULT Y X_d)
```

1.2 Theorems

[AND_BEFORE_ABSORB]

```
⊢ ∀ X Y Z.
      D_AND (D_AND (D_BEFORE X Y) (D_BEFORE Y Z))
        (D_BEFORE X Z) =
      D_AND (D_BEFORE X Y) (D_BEFORE Y Z)
```

[AND_INTER]

```
⊢ ∀ p t X Y.
      DFT_event p (D_AND X Y) t =
      DFT_event p X t ∩ DFT_event p Y t
```

[BEFORE_event_GT0]

```
⊢ ∀X Y p t.
      (∀s. 0 ≤ X s) ⇒
      (DFT_event p (D_BEFORE X Y) t =
       {s | X s ≤ Normal t ∧ 0 ≤ X s ∧ X s < Y s} ∩ p_space p)
```

[BEFORE_PREIMAGE_event_GT0]

```
⊢ ∀X Y p t.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ⇒
      (DFT_event p (D_BEFORE X Y) t =
       PREIMAGE (λs. (real (X s),real (Y s)))
         {(w,u) | 0 ≤ w ∧ w < u ∧ w ≤ t} ∩ p_space p)
```

[CDF_prob_DFT_event]

```
⊢ ∀X p t.
      (∀s. X s ≠ PosInf ∧ 0 ≤ X s) ⇒
      (CDF p (λs. real (X s)) t = prob p (DFT_event p X t))
```

[D_ABSORB_AND]

```
⊢ ∀X Y. D_AND X (D_OR X Y) = X
```

[D_AND_ALWAYS]

```
⊢ ∀X. (∀s. 0 ≤ X s) ⇒ (D_AND X ALWAYS = X)
```

[D_AND_ASSOC]

```
⊢ ∀X Y Z. D_AND (D_AND X Y) Z = D_AND X (D_AND Y Z)
```

[D_AND_COMM]

```
⊢ ∀X Y. D_AND X Y = D_AND Y X
```

[D_AND_ID]

```
⊢ ∀X. D_AND X X = X
```

[D_AND_NEVER]

```
⊢ ∀X. D_AND X NEVER = NEVER
```

[D_AND_OVER_OR]

```
⊢ ∀X Y Z. D_AND X (D_OR Y Z) = D_OR (D_AND X Y) (D_AND X Z)
```

[D_AND_SIMULT_ABSORB]

```
⊢ ∀X Y Z.
      D_AND (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
        (D_SIMULT X Z) =
      D_AND (D_SIMULT X Y) (D_SIMULT Y Z)
```

[D_BEFORE_NON_ASSOC1]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_BEFORE Y Z) =
      D_OR (D_BEFORE X Y)
        (D_AND (D_AND X Y) (D_OR (D_BEFORE Z Y) (D_SIMULT Z Y)))
```

[D_BEFORE_NON_ASSOC2]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_BEFORE Y Z) =
      D_OR (D_BEFORE X Y)
        (D_AND (D_AND X Y) (D_INCLUSIVE_BEFORE Z Y))
```

[D_BEFORE_NON_ASSOC3]

```
⊢ ∀X Y Z.
      D_BEFORE (D_BEFORE X Y) Z =
      D_AND (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_BEFORE_NON_COMM]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_BEFORE Y X) = NEVER
```

[D_BEFORE_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_INCLUSIVE_BEFORE Y Z) =
      D_OR (D_BEFORE X Y) (D_AND (D_AND X Y) (D_BEFORE Z Y))
```

[D_BEFORE_OVER_SIMULT]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_SIMULT Y Z) =
      D_OR
        (D_OR
           (D_OR (D_AND X (D_BEFORE Y Z))
              (D_AND X (D_BEFORE Z Y))) (D_BEFORE X Y))
        (D_BEFORE X Z)
```

[D_INCLUSIVE_AND]

```
⊢ ∀X Y.
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) =
      D_SIMULT X Y
```

[D_INCLUSIVE_BEFORE_def2]

```
⊢ ∀X Y.
      D_INCLUSIVE_BEFORE X Y =
      D_OR (D_BEFORE X Y) (D_SIMULT X Y)
```

[D_INCLUSIVE_ID]

```
⊢ ∀X. D_INCLUSIVE_BEFORE X X = X
```

[D_INCLUSIVE_OR_ABSORB]

```
⊢ ∀X Y.
      D_OR (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) =
      D_OR X Y
```

[D_LEFT_AND_OVER_INCLUSIVE]

```
⊢ ∀X Y.
      D_AND X (D_INCLUSIVE_BEFORE X Y) = D_INCLUSIVE_BEFORE X Y
```

[D_LEFT_AND_OVER_SIMULT]

```
⊢ ∀X Y. D_AND X (D_SIMULT X Y) = D_SIMULT X Y
```

[D_LEFT_BEFORE_OVER_AND]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_AND Y Z) =
      D_OR (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_LEFT_BEFORE_OVER_OR]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_OR Y Z) =
      D_AND (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_AND]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_AND Y Z) =
      D_OR (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_BEFORE]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_BEFORE Y Z) =
      D_OR
        (D_OR (D_BEFORE X Y)
           (D_AND (D_AND X Y) (D_INCLUSIVE_BEFORE Z Y)))
        (D_AND (D_SIMULT X Y) (D_BEFORE Y Z))
```

[D_LEFT_INCLUSIVE_OVER_INCLUSIVE1]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_INCLUSIVE_BEFORE Y Z) =
      D_OR
        (D_OR (D_BEFORE X Y) (D_AND (D_AND X Y) (D_BEFORE Z Y)))
        (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z))
```

[D_LEFT_INCLUSIVE_OVER_OR]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_OR Y Z) =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_SIMULT]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_SIMULT Y Z) =
      D_OR
        (D_OR
           (D_OR
              (D_OR (D_AND X (D_BEFORE Y Z))
                 (D_AND X (D_BEFORE Z Y))) (D_BEFORE X Y))
           (D_BEFORE X Z))
        (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
```

[D_LEFT_OR_INCLUSIVE_ABSORB]

```
⊢ ∀X Y. D_OR X (D_INCLUSIVE_BEFORE X Y) = X
```

[D_LEFT_OR_OVER_INCLUSIVE]

```
⊢ ∀X Y. D_OR Y (D_INCLUSIVE_BEFORE X Y) = D_OR X Y
```

[D_LEFT_OR_OVER_SIMULT]

```
⊢ ∀X Y. D_OR X (D_SIMULT X Y) = X
```

[D_LEFT_SIMULT_OVER_AND1]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_AND Y Z) =
      D_OR
        (D_OR (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
           (D_AND (D_SIMULT X Y) (D_BEFORE Z Y)))
        (D_AND (D_SIMULT X Z) (D_BEFORE Y Z))
```

[D_LEFT_SIMULT_OVER_AND2]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_AND Y Z) =
      D_OR (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Z Y))
        (D_AND (D_SIMULT X Z) (D_INCLUSIVE_BEFORE Y Z))
```

[D_LEFT_SIMULT_OVER_BEFORE]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_BEFORE Y Z) =
      D_AND (D_SIMULT X Y) (D_BEFORE Y Z)
```

[D_LEFT_SIMULT_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_INCLUSIVE_BEFORE Y Z) =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[D_LEFT_SImULT_OVER_OR]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_OR Y Z) =
      D_OR
        (D_OR (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
           (D_AND (D_SIMULT X Y) (D_BEFORE Y Z)))
        (D_AND (D_SIMULT X Z) (D_BEFORE Z Y))
```

[D_LEFT_SIMULT_OVER_OR2]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_OR Y Z) =
      D_OR (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z))
        (D_AND (D_SIMULT X Z) (D_INCLUSIVE_BEFORE Z Y))
```

[D_OR_ALWAYS]

```
⊢ ∀X. (∀s. 0 ≤ X s) ⇒ (D_OR X ALWAYS = ALWAYS)
```

[D_OR_ASSOC]

```
⊢ ∀X Y Z. D_OR (D_OR X Y) Z = D_OR X (D_OR Y Z)
```

[D_OR_COMM]

```
⊢ ∀X Y. D_OR X Y = D_OR Y X
```

[D_OR_ID]

```
⊢ ∀X. D_OR X X = X
```

[D_OR_NEVER]

```
⊢ ∀X. D_OR X NEVER = X
```

[D_OR_OVER_AND]

```
⊢ ∀X Y Z. D_OR X (D_AND Y Z) = D_AND (D_OR X Y) (D_OR X Z)
```

[D_RIGHT_BEFORE_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_BEFORE (D_INCLUSIVE_BEFORE X Y) Z =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_BEFORE]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_BEFORE X Y) Z =
      D_AND (D_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_INCLUSIVE]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_INCLUSIVE_BEFORE X Y) Z =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT1]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT2]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT3]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_SIMULT (D_INCLUSIVE_BEFORE X Z)
        (D_INCLUSIVE_BEFORE Y Z)
```

[D_SIMULT_COMM]

```
⊢ ∀X Y. D_SIMULT X Y = D_SIMULT Y X
```

[D_SIMULT_ID]

```
⊢ ∀X. D_SIMULT X X = X
```

[D_SIMULT_NEVER]

```
⊢ ∀X. D_SIMULT X NEVER = NEVER
```

[DFT_BEFORE_event_GT0]

```
⊢ ∀X Y p t.
      (∀s. 0 ≤ X s) ⇒
      (DFT_event p (D_BEFORE X Y) t =
       PREIMAGE (λs. (X s,Y s))
         {(w,u) | 0 ≤ w ∧ w ≤ Normal t ∧ w < u} ∩ p_space p)
```

[DFT_event_AFTER_PREIMAGE_GT0]

```
⊢ ∀p t X Y.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ⇒
      (DFT_event p (D_AND Y (D_BEFORE X Y)) t =
       PREIMAGE (λx. (real (X x),real (Y x)))
         {(u,w) | u < w ∧ 0 ≤ w ∧ w ≤ t} ∩ p_space p)
```

[DFT_event_AND_BEFORE]

```
⊢ ∀p t X Y.
      DFT_event p (D_AND Y (D_BEFORE X Y)) t =
      {u | X u < Y u ∧ Y u ≤ Normal t} ∩ p_space p
```

[DFT_event_AND_BEFORE_PREIMAGE]

$\vdash \forall p\ t\ X\ Y.$
  DFT_event p (D_AND Y (D_BEFORE X Y)) t =
  PREIMAGE ($\lambda x.$ (X x,Y x)) $\{(u,w)$ | $u < w \land w \leq$ Normal $t\} \cap$
  p_space p

[DFT_event_AND_BEFORE_PREIMAGE_GT0]

$\vdash \forall p\ t\ X\ Y.$
  ($\forall s.\ 0 \leq Y\ s$) $\Rightarrow$
  (DFT_event p (D_AND Y (D_BEFORE X Y)) t =
   PREIMAGE ($\lambda x.$ (X x,Y x))
    $\{(u,w)$ | $u < w \land 0 \leq w \land w \leq$ Normal $t\} \cap$ p_space p)

[DFT_PREIMAGE]

$\vdash \forall X\ p\ t.$
  DFT_event p X t =
  PREIMAGE X $\{u$ | $u \leq$ Normal $t\} \cap$ p_space p

[FDEP_OR]

$\vdash \forall X\ Y.$ FDEP X Y = D_OR X Y

[HSP_AND]

$\vdash \forall X\ Y.$ HSP Y X = D_AND Y X

[IN_REST]

$\vdash \forall x\ s.\ x \in$ REST $s \iff x \in s \land x \neq$ CHOICE s

[IN_UNIONL]

$\vdash \forall l\ v.\ v \in$ UNIONL $l \iff \exists s.$ MEM s l $\land v \in s$

[INCLUSIVE_BEFORE_NEVER]

$\vdash \forall X.$ D_INCLUSIVE_BEFORE X NEVER = X

[lborel_le]

$\vdash \forall x.\ \{w$ | $w \leq x\} \in$ measurable_sets lborel

[LEFT_AND_OVER_BEFORE_ABSORB]

$\vdash \forall X\ Y.$ D_AND X (D_BEFORE X Y) = D_BEFORE X Y

[LEFT_OR_OVER_BEFORE_ABSORB]

$\vdash \forall X\ Y.$ D_OR X (D_BEFORE X Y) = X

[Lemma_64]

$\vdash \forall X\ Y.$
  D_OR (D_AND X (D_INCLUSIVE_BEFORE Y X))
    (D_AND Y (D_INCLUSIVE_BEFORE X Y)) =
  D_AND X Y

[Lemma_65]

```
⊢ ∀X Y.
       D_OR (D_INCLUSIVE_BEFORE X Y)
         (D_AND X (D_INCLUSIVE_BEFORE Y X)) =
       X
```

[Lemma_66]

```
⊢ ∀X Y Z.
       D_AND
         (D_AND (D_INCLUSIVE_BEFORE X Y)
            (D_INCLUSIVE_BEFORE Y Z)) (D_INCLUSIVE_BEFORE X Z) =
       D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[Lemma_67]

```
⊢ ∀X Y.
       D_OR (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Y) =
       D_INCLUSIVE_BEFORE X Y
```

[Lemma_68]

```
⊢ ∀X Y.
       D_OR (D_INCLUSIVE_BEFORE X Y) (D_SIMULT X Y) =
       D_INCLUSIVE_BEFORE X Y
```

[Lemma_69]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_SIMULT X Y) = NEVER
```

[Lemma_70]

```
⊢ ∀X Y Z.
       D_AND (D_BEFORE X Y) (D_SIMULT Y Z) =
       D_AND (D_BEFORE X Z) (D_SIMULT Y Z)
```

[Lemma_71]

```
⊢ ∀X Y.
       D_AND (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Y) =
       D_BEFORE X Y
```

[Lemma_72]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) = NEVER
```

[Lemma_73]

```
⊢ ∀X Y.
       D_AND (D_INCLUSIVE_BEFORE X Y) (D_SIMULT X Y) =
       D_SIMULT X Y
```

[Lemma_74]

```
⊢ ∀X Y.
       D_OR (D_OR (D_BEFORE X Y) (D_SIMULT X Y)) (D_BEFORE Y X) =
       D_OR X Y
```

[Lemma_75]

```
⊢ ∀X Y.
    D_OR (D_OR (D_AND X (D_BEFORE Y X)) (D_SIMULT X Y))
      (D_AND Y (D_BEFORE X Y)) =
    D_AND X Y
```

[Lemma_76]

```
⊢ ∀X Y.
    D_OR (D_OR (D_BEFORE X Y) (D_SIMULT X Y))
      (D_AND X (D_BEFORE Y X)) =
    X
```

[Lemma_77]

```
⊢ ∀X Y Z.
    D_AND (D_AND (D_BEFORE X Y) (D_BEFORE Y Z))
      (D_INCLUSIVE_BEFORE X Z) =
    D_AND (D_BEFORE X Y) (D_BEFORE Y Z)
```

[NEVER_INCLUSIVE_BEFORE_BEFORE]

```
⊢ ∀X. D_INCLUSIVE_BEFORE NEVER X = NEVER
```

[normal_real_mul]

```
⊢ ∀x y.
    x ≠ PosInf ∧ x ≠ NegInf ∧ y ≠ PosInf ∧ y ≠ NegInf ⇒
    (Normal (real x × real y) =
     Normal (real x) × Normal (real y))
```

[OR_UNION]

```
⊢ ∀p t X Y.
    DFT_event p (D_OR X Y) t =
    DFT_event p X t ∪ DFT_event p Y t
```

[PAND_DEF2]

```
⊢ ∀X Y. P_AND X Y = D_AND Y (D_INCLUSIVE_BEFORE X Y)
```

[PIE_lem2_or]

```
⊢ ∀f X Y.
    ALL_DISTINCT [X; Y] ∧
    (∀x. x ∈ {{X}; {Y}; {X; Y}} ⇒ f x ≠ PosInf) ⇒
    (SIGMA f {{X}; {Y}; {X; Y}} = f {X} + f {Y} + f {X; Y})
```

[PREIMAGE_EXTREAL_REAL]

```
⊢ ∀X t p.
    (∀s. X s ≠ PosInf ∧ 0 ≤ X s) ⇒
    (PREIMAGE X {u | u ≤ Normal t} ∩ p_space p =
     PREIMAGE (λs. real (X s)) {u | u ≤ t} ∩ p_space p)
```

[PREIMAGE_EXTREAL_REAL_2RV]

```
⊢ ∀X Y t p.
      (∀s. X s ≠ PosInf ∧ 0 ≤ X s ∧ Y s ≠ PosInf ∧ 0 ≤ Y s) ∧
      0 ≤ t ⇒
      (PREIMAGE (λs. (X s,Y s))
         {(u,w) | u < w ∧ 0 ≤ w ∧ w ≤ Normal t} ∩ p_space p =
       PREIMAGE (λs. (real (X s),real (Y s)))
         {(u,w) | u < w ∧ 0 ≤ w ∧ w ≤ t} ∩ p_space p)
```

[PREIMAGE_EXTREAL_REAL_2RV_BEFORE]

```
⊢ ∀X Y t p.
      (∀s. X s ≠ PosInf ∧ 0 ≤ X s ∧ Y s ≠ PosInf ∧ 0 ≤ Y s) ∧
      0 ≤ t ⇒
      (PREIMAGE (λs. (X s,Y s))
         {(w,u) | 0 ≤ w ∧ w < u ∧ w ≤ Normal t} ∩ p_space p =
       PREIMAGE (λs. (real (X s),real (Y s)))
         {(w,u) | 0 ≤ w ∧ w < u ∧ w ≤ t} ∩ p_space p)
```

[PROB_AND]

```
⊢ ∀p t X Y.
      rv_gt0_ninfty [X; Y] ∧
      indep_var p lborel (λs. real (X s)) lborel
        (λs. real (Y s)) ⇒
      (prob p (DFT_event p (D_AND X Y) t) =
       CDF p (λs. real (X s)) t × CDF p (λs. real (Y s)) t)
```

[PROB_DFT_AFTER]

```
⊢ ∀X Y p fy t.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ∧ prob_space p ∧
      indep_var p lborel (λs. real (X s)) lborel
        (λs. real (Y s)) ∧
      distributed p lborel (λs. real (Y s)) fy ∧
      (∀y. 0 ≤ fy y) ∧ cont_CDF p (λs. real (X s)) ∧
      measurable_CDF p (λs. real (X s)) ⇒
      (prob p (DFT_event p (D_AND Y (D_BEFORE X Y)) t) =
       pos_fn_integral lborel
         (λy.
              fy y ×
              (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
               CDF p (λs. real (X s)) y)))
```

[PROB_DFT_BEFORE]

```
⊢ ∀X Y p fx t.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ∧ prob_space p ∧
      indep_var p lborel (λs. real (X s)) lborel
        (λs. real (Y s)) ∧
      distributed p lborel (λs. real (X s)) fx ∧
      (∀x. 0 ≤ fx x) ∧ measurable_CDF p (λs. real (Y s)) ⇒
```

```
(prob p (DFT_event p (D_BEFORE X Y) t) =
 pos_fn_integral lborel
   (λx.
        fx x ×
        (indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
         (1 - CDF p (λs. real (Y s)) x))))
```

[PROB_OR]

```
⊢ ∀X Y p t.
      rv_gt0_ninfty [X; Y] ∧ All_distinct_events p [X; Y] t ∧
      indep_var p lborel (λs. real (X s)) lborel
        (λs. real (Y s)) ⇒
      (prob p (DFT_event p (D_OR X Y) t) =
       CDF p (λs. real (X s)) t + CDF p (λs. real (Y s)) t -
       CDF p (λs. real (X s)) t × CDF p (λs. real (Y s)) t)
```

[RIGHT_BEFORE_OVER_AND]

```
⊢ ∀X Y Z.
      D_BEFORE (D_AND X Y) Z =
      D_AND (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_OR]

```
⊢ ∀X Y Z.
      D_BEFORE (D_OR X Y) Z =
      D_OR (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_SIMULT1]

```
⊢ ∀X Y Z.
      D_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_BEFORE X Z)
```

[RIGHT_BEFORE_OVER_SIMULT2]

```
⊢ ∀X Y Z.
      D_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_SIMULT3]

```
⊢ ∀X Y Z.
      D_BEFORE (D_SIMULT X Y) Z =
      D_SIMULT (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_OR_OVER_BEFORE_ABSORB]

```
⊢ ∀X Y. D_OR (D_BEFORE X Y) Y = D_OR X Y
```